

Dynamics on geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond

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Abstract. We present recent results on counting and distribution of circles in a given circle packing invariant under a geometrically finite Kleinian group and discuss how the dynamics of flows on geometrically finite hyperbolic 3 manifolds are related. Our results apply to Apollonian circle packings, Sierpinski curves, Schottky dances, etc.

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1. Introduction

Let G be a connected semisimple Lie group and $\Gamma < G$ a discrete subgroup with finite co-volume. Dynamics of flows on the homogeneous space $\Gamma \backslash G$ have been studied intensively over the last several decades and brought many surprising applications in various fields notably including analytic number theory, arithmetic geometry and Riemannian geometry (see [45], [58], [12], [18], [32], [78], [41], [74], [16], [21], [22], [76], [15], [49], [33], [75], [25], [27], [26], [66], etc.) The assumption that the volume of $\Gamma \backslash G$ is finite is crucial in most developments in the ergodic theory for flows on $\Gamma \backslash G$, as many basic ergodic theorems fail in the setting of an infinite measure space. It is unclear what kind of measure theoretic and topological rigidity for flows on $\Gamma \backslash G$ can be expected for a general discrete subgroup Γ .

In this article we consider the situation when G is the isometry group of the real hyperbolic space \mathbb{H}^n , $n \geq 2$, and $\Gamma < G$ is a geometrically finite discrete subgroup. In such cases we have a rich theory of the Patterson-Sullivan density and the structure of a fundamental domain for Γ in \mathbb{H}^n is well understood. Using these we obtain certain equidistribution results for specific flows on the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^n)$ and apply them to prove results on counting and equidistribution for circles in a given circle packing of the plane (and also of the sphere) invariant under geometrically finite groups.

There are numerous natural questions which arise from the analogy with the finite volume cases and most of them are unsolved. We address some of them in

the last section. We remark that an article by Sarig [61] discusses related issues but for geometrically *infinite* surfaces.

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2. Preliminaries

We review some of basic definitions as well as set up notations. Let G be the identity component of the isometry group of the real hyperbolic space \mathbb{H}^n , $n \geq 2$. Let $\Gamma < G$ be a torsion-free discrete subgroup. We denote by $\partial_\infty(\mathbb{H}^n)$ the geometric boundary of \mathbb{H}^n . The limit set $\Lambda(\Gamma)$ of Γ is defined to be the set of accumulation points of an orbit of Γ in $\mathbb{H}^n \cup \partial_\infty(\mathbb{H}^n)$. As Γ acts on \mathbb{H}^n properly discontinuously, $\Lambda(\Gamma)$ lies in $\partial_\infty(\mathbb{H}^n)$. Its complement $\Omega(\Gamma) := \partial_\infty(\mathbb{H}^n) - \Lambda(\Gamma)$ is called the domain of discontinuity for Γ .

An element $g \in G$ is called parabolic if it fixes a unique point in $\partial_\infty(\mathbb{H}^n)$ and loxodromic if it fixes two points in $\partial_\infty(\mathbb{H}^n)$. A limit point $\xi \in \Lambda(\Gamma)$ is called a parabolic fixed point if it is fixed by a parabolic element of Γ and called a radial limit point (or a conical limit point or a point of approximation) if for some geodesic ray β tending to ξ and some point $x \in \mathbb{H}^n$, there is a sequence $\gamma_i \in \Gamma$ with $\gamma_i x \rightarrow \xi$ and $d(\gamma_i x, \beta)$ is bounded, where d denotes the hyperbolic distance. A parabolic fixed point ξ is called bounded if $\text{Stab}_\Gamma(\xi) \backslash (\Lambda(\Gamma) - \{\xi\})$ is compact.

The convex core C_Γ of Γ is defined to be the minimal convex set in \mathbb{H}^n mod Γ which contains all geodesics connecting any two points in $\Lambda(\Gamma)$. A discrete subgroup Γ is called *geometrically finite* if the unit neighborhood of its convex core has finite volume and called *convex co-compact* if its convex core is compact. It is clear that a (resp. co-compact) lattice in G is geometrically finite (resp. convex co-compact). Bowditch showed [5] that Γ is geometrically finite if and only if $\Lambda(\Gamma)$ consists entirely of radial limit points and bounded parabolic fixed points. It is further equivalent to saying that Γ is finitely generated for $n = 2$, and that Γ admits a finite sided fundamental domain in \mathbb{H}^3 for $n = 3$. We refer to [5] for other equivalent definitions.

Γ is called *elementary* if $\Lambda(\Gamma)$ consists of at most two points, or equivalently, Γ has an abelian subgroup of finite index.

We denote by $0 \leq \delta_\Gamma \leq n - 1$ the critical exponent of Γ , that is, the abscissa of convergence of the Poincare series of Γ :

$$\mathcal{P}_\Gamma(s) := \sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}$$

where $o \in \mathbb{H}^n$. For a non-elementary group Γ , δ_Γ is positive and Sullivan [71] showed that for Γ geometrically finite, δ_Γ is equal to the Hausdorff dimension of the limit set $\Lambda(\Gamma)$.

For $\xi \in \partial_\infty(\mathbb{H}^n)$ and $y_1, y_2 \in \mathbb{H}^n$, the Busemann function $\beta_\xi(y_1, y_2)$ measures a signed distance between horospheres passing through y_1 and y_2 based at ξ :

$$\beta_\xi(y_1, y_2) = \lim_{t \rightarrow \infty} d(y_1, \xi_t) - d(y_2, \xi_t)$$

where ξ_t is a geodesic ray toward ξ .

For a vector u in the unit tangent bundle $T^1(\mathbb{H}^n)$, we define $u^\pm \in \partial_\infty(\mathbb{H}^n)$ to be the two end points of the geodesic determined by u :

$$u^+ := \lim_{t \rightarrow \infty} g^t(u) \quad \text{and} \quad u^- := \lim_{t \rightarrow -\infty} g^t(u)$$

where $\{g^t\}$ denotes the geodesic flow.

We denote by $\pi : T^1(\mathbb{H}^n) \rightarrow \mathbb{H}^n$ the canonical projection. Fixing a base point $o \in \mathbb{H}^n$, the map

$$u \mapsto (u^+, u^-, \beta_{u^-}(\pi(u), o))$$

yields a homeomorphism between $T^1(\mathbb{H}^n)$ and $(\partial_\infty(\mathbb{H}^n) \times \partial_\infty(\mathbb{H}^n) - \{(\xi, \xi) : \xi \in \partial_\infty(\mathbb{H}^n)\}) \times \mathbb{R}$.

Throughout the paper we assume that Γ is non-elementary.

Patterson-Sullivan density: Generalizing the work of Patterson [55] for $n = 2$, Sullivan [71] constructed a Γ -invariant conformal density $\{\nu_x : x \in \mathbb{H}^n\}$ of dimension δ_Γ on $\Lambda(\Gamma)$. That is, each ν_x is a finite Borel measure on $\partial_\infty(\mathbb{H}^n)$ supported on $\Lambda(\Gamma)$ satisfying that for any $x, y \in \mathbb{H}^n$, $\xi \in \partial_\infty(\mathbb{H}^n)$ and $\gamma \in \Gamma$,

$$\gamma_* \nu_x = \nu_{\gamma x} \quad \text{and} \quad \frac{d\nu_y}{d\nu_x}(\xi) = e^{-\delta_\Gamma \beta_\xi(y, x)},$$

where $\gamma_* \nu_x(R) = \nu_x(\gamma^{-1}(R))$.

For Γ geometrically finite, such conformal density $\{\nu_x\}$ exists uniquely up to homothety. In fact, fixing $o \in \mathbb{H}^n$, $\{\nu_x\}$ is a constant multiple of the following family $\{\nu_{x,o}\}$ where $\nu_{x,o}$ is the weak-limit as $s \rightarrow \delta_\Gamma^+$ of the family of measures

$$\nu_{x,o}(s) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(o, \gamma o)}} \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma o)} \delta_{\gamma o}$$

where $\delta_{\gamma o}$ denotes the Dirac measure at γo .

Consider the Laplacian Δ on \mathbb{H}^n . In the upper half-space coordinates $\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, y) : y > 0\}$ with the metric $\frac{\sqrt{dx_1^2 + \dots + dx_{n-1}^2 + dy^2}}{y}$, it is given as

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial y^2} \right) + (n-2)y \frac{\partial}{\partial y}$$

(strictly speaking, this is the negative of the usual hyperbolic Laplacian). Sullivan [71] showed that

$$\phi_\Gamma(x) := |\nu_x|$$

is an eigenfunction for Δ with eigenvalue $\delta_\Gamma(n-1-\delta_\Gamma)$. From the Γ -invariance of the Patterson-Sullivan density $\{\nu_x\}$, ϕ_Γ is a function on $\Gamma \backslash \mathbb{H}^n$. Sullivan further showed that if Γ is geometrically finite and $\delta_\Gamma > (n-1)/2$, ϕ_Γ belongs to $L^2(\Gamma \backslash \mathbb{H}^n)$ and is a unique (up to a constant multiple) positive eigenfunction with the smallest eigenvalue $\delta_\Gamma(n-1-\delta_\Gamma)$ (cf. [73]). Combined with a result of Yau [77], it follows that $\delta_\Gamma = n-1$ if and only if Γ is a lattice in G .

Bowen-Margulis-Sullivan measure: Fixing the Patterson-Sullivan density $\{\nu_x\}$, the Bowen-Margulis-Sullivan measure m_Γ^{BMS} ([6], [46], [72]) is the induced measure on $T^1(\Gamma \backslash \mathbb{H}^n)$ of the following Γ -invariant measure on $T^1(\mathbb{H}^n)$:

$$d\tilde{m}^{\text{BMS}}(u) = e^{\delta_\Gamma \beta_{u^+}(x, \pi(u))} e^{\delta_\Gamma \beta_{u^-}(x, \pi(u))} d\nu_x(u^+) d\nu_x(u^-) dt$$

where $x \in \mathbb{H}^n$.

It follows from the conformality of $\{\nu_x\}$ that this definition is independent of the choice of x . The measure m_Γ^{BMS} is invariant under the geodesic flow and is supported on the non-wandering set $\{u \in T^1(\Gamma \backslash \mathbb{H}^n) : u^\pm \in \Lambda(\Gamma)\}$ of the geodesic flow. Sullivan showed that for Γ geometrically finite, the total mass $|m_\Gamma^{\text{BMS}}|$ is finite and the geodesic flow is ergodic with respect to m_Γ^{BMS} [72]. This is a very important point for the ergodic theory on geometrically finite hyperbolic manifolds, since despite of the fact that the Liouville measure is infinite, we do have a finite measure on $T^1(\Gamma \backslash \mathbb{H}^n)$ which is invariant and ergodic for the geodesic flow. Rudolph [60] showed that the geodesic flow is even mixing with respect to m_Γ^{BMS} .

3. Counting and distribution of circles in the plane

A circle packing in the plane \mathbb{C} is simply a union of circles. As circles may intersect with each other beyond tangency points, our definition of a circle packing is more general than what is usually thought of. For a given circle packing \mathcal{P} in the plane, we discuss questions on counting and distribution of small circles in \mathcal{P} . A natural size of a circle is measured by its radius. We will use the curvature (=the reciprocal of the radius) of a circle instead.

We suppose that \mathcal{P} is infinite and that \mathcal{P} is locally finite in the sense that for any $T > 0$, there are only finitely many circles of curvature at most T in any fixed bounded region of the plane. See Fig. 1, 6 and 8 for examples of locally finite packings.

For a bounded region E in the plane \mathbb{C} , we consider the following counting function:

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \text{Curv}(C) < T\}$$

where $\text{Curv}(C)$ denotes the curvature of C . The local finiteness assumption is so that $N_T(\mathcal{P}, E) < \infty$ for any bounded E . We ask if there is an asymptotic for

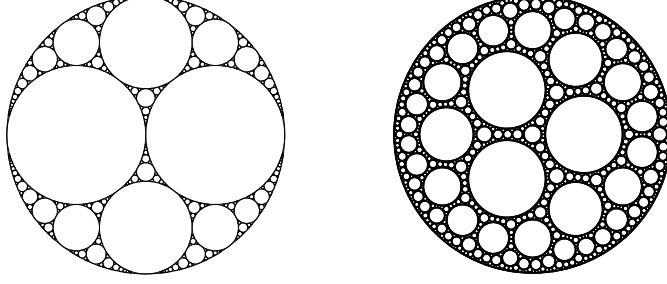


Figure 1. Apollonian circle packing and Sierpinski curve (by C. McMullen)

$N_T(\mathcal{P}, E)$ as T tends to infinity and what the dependence of such an asymptotic on E is, if exists.

Consider the upper half space model $\mathbb{H}^3 = \{(z, r) : z \in \mathbb{C}, r > 0\}$ with the hyperbolic metric given by $\frac{\sqrt{|dz|^2 + dr^2}}{r}$. An elementary but helpful observation is that if we denote by $\hat{C} \subset \mathbb{H}^3$ the convex hull of C , that is, the northern hemisphere above C , then $N_T(\mathcal{P}, E)$ is equal to the number of hemispheres of height at most T^{-1} in \mathbb{H}^3 whose boundaries lie in \mathcal{P} and intersect E , as the radius of a circle is same as the height of the corresponding hemisphere.

Let $\Gamma < \text{PSL}_2(\mathbb{C})$ be a geometrically finite discrete subgroup and fix a Γ -invariant Patterson-Sullivan density $\{\nu_x : x \in \mathbb{H}^3\}$.

In order to present our theorem on the asymptotic of $N_T(\mathcal{P}, E)$ for \mathcal{P} invariant under Γ , we introduce two new invariants associated to Γ and \mathcal{P} . The first one is a Borel measure on \mathbb{C} which depends only on Γ .

Definition 3.1. Define a Borel measure ω_Γ on \mathbb{C} : for $\psi \in C_c(\mathbb{C})$,

$$\omega_\Gamma(\psi) = \int_{z \in \mathbb{C}} \psi(z) e^{\delta_\Gamma \beta_z(x, z+j)} d\nu_x(z)$$

where $j = (0, 1) \in \mathbb{H}^3$ and $x \in \mathbb{H}^3$. By the conformal property of $\{\nu_x\}$, this definition is independent of the choice of $x \in \mathbb{H}^3$.

Note that ω_Γ is supported on $\Lambda(\Gamma) \cap \mathbb{C}$ and in particular that $\omega_\Gamma(E) > 0$ if the interior of E intersects $\Lambda(\Gamma) \cap \mathbb{C}$ non-trivially. We compute:

$$d\omega_\Gamma = (|z|^2 + 1)^{\delta_\Gamma} d\nu_j.$$

The second one is a number in $[0, \infty]$ measuring certain size of \mathcal{P} :

Definition 3.2 (The Γ -skinning size of \mathcal{P}). For a circle packing \mathcal{P} invariant under Γ , we define:

$$\text{sk}_\Gamma(\mathcal{P}) := \sum_{i \in I} \int_{s \in \text{Stab}_\Gamma(C_i^\dagger) \setminus C_i^\dagger} e^{\delta_\Gamma \beta_{s^+}(x, \pi(s))} d\nu_x(s^+)$$

where $x \in \mathbb{H}^3$, $\{C_i : i \in I\}$ is a set of representatives of Γ -orbits in \mathcal{P} and $C_i^\dagger \subset T^1(\mathbb{H}^3)$ is the set of unit normal vectors to the convex hull \hat{C}_i of C_i . Again by the conformal property of $\{\nu_x\}$, the definition of $\text{sk}_\Gamma(\mathcal{P})$ is independent of the choice of x and the choice of representatives $\{C_i\}$.

We remark that the value of $\text{sk}_\Gamma(\mathcal{P})$ can be zero or infinite in general and we do not assume any condition on $\text{Stab}_\Gamma(C_i^\dagger)$'s (they may even be trivial). By the interior of a circle C , we mean the open disk which is enclosed by C . We then have the following:

Theorem 3.3 ([51]). *Let Γ be a non-elementary geometrically finite discrete subgroup of $\text{PSL}_2(\mathbb{C})$ and let $\mathcal{P} = \cup_{i \in I} \Gamma(C_i)$ be an infinite, locally finite, and Γ -invariant circle packing with finitely many Γ -orbits.*

Suppose one of the following conditions hold:

1. Γ is convex co-compact;
2. all circles in \mathcal{P} are mutually disjoint;
3. $\cup_{i \in I} C_i^\circ \subset \Omega(\Gamma)$ where C_i° denotes the interior of C_i .

For any bounded region E of \mathbb{C} whose boundary is of zero Patterson-Sullivan measure, we have

$$N_T(\mathcal{P}, E) \sim \frac{\text{sk}_\Gamma(\mathcal{P})}{\delta_\Gamma \cdot |m_\Gamma^{\text{BMS}}|} \cdot \omega_\Gamma(E) \cdot T^{\delta_\Gamma} \quad \text{as } T \rightarrow \infty$$

and $0 < \text{sk}_\Gamma(\mathcal{P}) < \infty$.

Remark 3.4. 1. If Γ is Zariski dense in $\text{PSL}_2(\mathbb{C})$, considered as a real algebraic group, any real algebraic curve has zero Patterson-Sullivan measure [23, Cor. 1.4]. Hence the above theorem applies to any Borel subset E whose boundary is a countable union of real algebraic curves.

2. We call the complement in $\hat{\mathbb{C}}$ of the set $\cup_{i \in I} \Gamma(C_i^\circ)$ the residual set of \mathcal{P} . The condition (3) above is then equivalent to saying that $\Lambda(\Gamma)$ is contained in the residual set of \mathcal{P} .
3. If we denote by $H_\infty^-(j)$ the contracting horosphere based at ∞ in $T^1(\mathbb{H}^3)$ which consists of all upward normal unit vectors on $\mathbb{C} + j = \{(z, 1) : z \in \mathbb{C}\}$, we can alternative write the measure ω_Γ as follows:

$$\omega_\Gamma(\psi) = \int_{u \in H_\infty^-(j)} \psi(u^-) e^{\delta_\Gamma \beta_{u^-}(x, \pi(u))} d\nu_x(u^-)$$

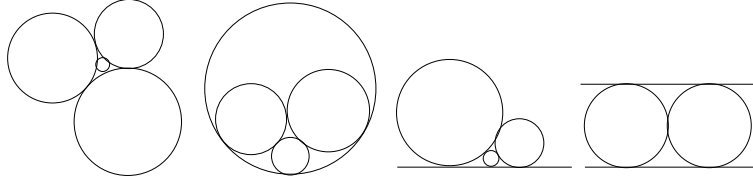


Figure 2. Possible configurations of four mutually tangent circles

and recognize that ω_Γ is the projection of the conditional of the Bowen-Margulis-Sullivan measure \tilde{m}^{BMS} on the horosphere $H_\infty^-(j)$ to \mathbb{C} via the map $u \mapsto u^-$. It is worthwhile to note that the hyperbolic metric on $\mathbb{C} + j$ is precisely the Euclidean metric.

4. Suppose that circles in \mathcal{P} are disjoint possibly except for tangency points and that $\Lambda(\Gamma)$ is equal to the residual set of \mathcal{P} . If ∞ is either in $\Omega(\Gamma)$ (that is, \mathcal{P} is bounded) or a parabolic fixed point for Γ , then δ_Γ is equal to the circle packing exponent $e_{\mathcal{P}}$ given by

$$e_{\mathcal{P}} = \inf\left\{s : \sum_{C \in \mathcal{P}} r(C)^s < \infty\right\} = \sup\left\{s : \sum_{C \in \mathcal{P}} r(C)^s = \infty\right\}$$

where $r(C)$ denotes the radius of C [54]. This extends the earlier work of Boyd [7] on bounded Apollonian circle packings.

We discuss some concrete circle packings to which our theorem applies.

3.1. Apollonian circle packings in the plane. Apollonian circle packings are one of the most beautiful circle packings whose construction can be described in a very simple manner based on an old theorem of Apollonius (262-190 BC). It says that given three mutually tangent circles in the plane, there are exactly two circles which are tangent to all the three circles.

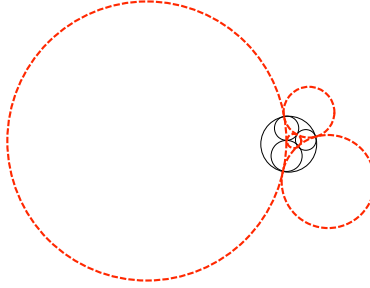


Figure 3. Dual circles

In order to construct an Apollonian circle packing, we start with four mutually tangent circles. See Fig. 2 for possible configurations. By Apollonius' theorem, there are precisely four new circles that are tangent to three of the four circles. Continuing to repeatedly add new circles tangent to three of the circles from the previous generations, we arrive at an infinite circle packing, called an Apollonian circle packing,

See Fig. 4 and 8 for examples of Apollonian circle packings where each circle is labeled by its curvature (that is, the reciprocal of its radius). There are also Apollonian packings which spread all over the plane as well as spread all over to the half plane. As circles in these packings would become enormously large after a few first generations, it is harder to draw them on paper.

There are many natural questions about Apollonian circle packings either from the number theoretic or the geometric point of view and we refer to the series of papers by Graham, Lagarias, Mallows, Wilks, and Yan especially [30] [29], and [17] as well as the letter of Sarnak to Lagarias [64] which inspired the author to work on the topic personally. Also see a more recent article [62].

To find the symmetry group of a given Apollonian packing \mathcal{P} , we consider the dual circles to any fixed four mutually tangent circles (see Fig. 3 where the red dotted circles are the dual circles to the black solid circles). Inversion with respect to each dual circle fixes three circles that the dual circle crosses perpendicularly and interchanges two circles tangent to those three circles. Hence the group, say, $\Gamma(\mathcal{P})$, generated by the four inversions with respect to the dual circles preserves the packing \mathcal{P} and there are four $\Gamma(\mathcal{P})$ orbits of circles in \mathcal{P} .

As the fundamental domain of $\Gamma(\mathcal{P})$ in \mathbb{H}^3 can be taken to be the exterior of the four hemispheres above the dual circles in \mathbb{H}^3 , $\Gamma(\mathcal{P})$ is geometrically finite. It is known that the limit set of $\Gamma(\mathcal{P})$ coincides precisely with the residual set of \mathcal{P} and hence the critical exponent of $\Gamma(\mathcal{P})$ is equal to the Hausdorff dimension of the residual set of \mathcal{P} , which is approximately

$$\alpha = 1.30568(8)$$

due to C. McMullen [48] (note that as any two Apollonian packings are equivalent to each other by a Möbius transformation, α is independent of \mathcal{P}). In particular it

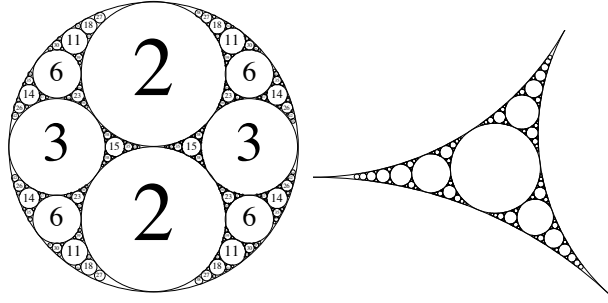


Figure 4. A bounded Apollonian circle packing and the Apollonian packing of a triangular region

follows that $\Gamma(\mathcal{P})$ is Zariski dense in the real algebraic group $\mathrm{PSL}_2(\mathbb{C})$ and hence we deduce the following from Theorem 3.3 and the remark following it:

Corollary 3.5 ([51]). *Let \mathcal{P} be an Apollonian circle packing. For any bounded region E of \mathbb{C} whose boundary is a countable union of real algebraic curves, we have*

$$N_T(\mathcal{P}, E) \sim \frac{\mathrm{sk}_{\Gamma_{\mathcal{P}}}(\mathcal{P})}{\alpha \cdot |m_{\Gamma_{\mathcal{P}}}^{\mathrm{BMS}}|} \cdot \omega_{\Gamma_{\mathcal{P}}}(E) \cdot T^\alpha \quad \text{as } T \rightarrow \infty$$

where $\Gamma_{\mathcal{P}} := \Gamma(\mathcal{P}) \cap \mathrm{PSL}_2(\mathbb{C})$.

- Remark 3.6.**
1. In the cases when \mathcal{P} is bounded and E is the largest disk in such \mathcal{P} , and when \mathcal{P} lies between two parallel lines and E is the whole period (see Fig. 8), the above asymptotic was previously obtained in [37] with a less explicit description of the main term.
 2. Corollary 3.5 applies to any triangular region \mathcal{T} (see Fig. 4) of an Apollonian circle packing.

3.2. More circle packings.

3.2.1. Counting circles in the limit set $\Lambda(\Gamma)$. If $\Gamma \backslash \mathbb{H}^3$ is a hyperbolic 3 manifold with boundary being totally geodesic, then Γ is automatically geometrically finite [34] and $\Omega(\Gamma)$ is a union of countably many disjoint open disks. Hence Theorem 3.3 applies to counting these open disks in $\Omega(\Gamma)$ with respect to the curvature, provided there are infinitely many such. The picture of a Sierpinski curve in Fig. 1 is a special case of this (so are Apollonian circle packings). More precisely, if Γ denotes the group generated by reflections in the sides of a unique regular tetrahedron whose convex core is bounded by four $\frac{\pi}{4}$ triangles and by four right hexagons, then the residual set of a Sierpinski curve in Fig. 1 coincides with $\Lambda(\Gamma)$ (see [47] for details), and it is known to be homeomorphic to the well-known Sierpinski carpet by a theorem of Claytor [9].

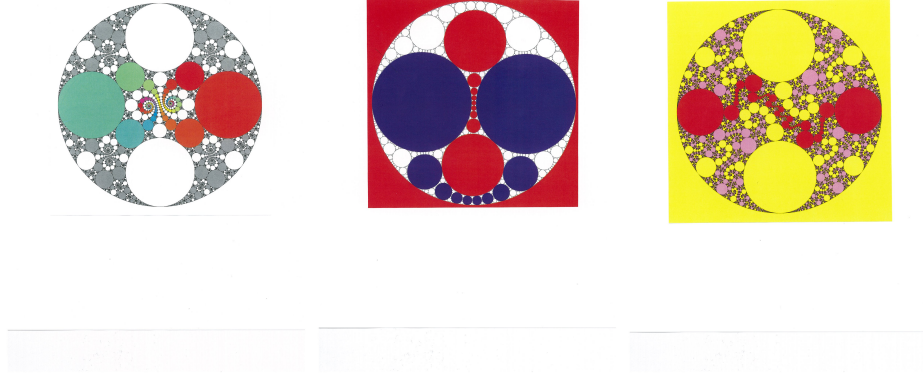


Figure 5. Limit sets of Schottky groups (reproduced with permission from *Indra's Pearls*, by D.Mumford, C. Series and D. Wright, copyright Cambridge University Press 2002).

Three pictures in Fig. 5 can be found in the beautiful book *Indra's pearls* by Mumford, Series and Wright [50] and the residual sets are the limit sets of some (geometrically finite) Schottky groups and hence our theorem applies to counting circles in those pictures.

3.2.2. Schottky dance. Other kinds of examples are obtained by considering the images of Schottky disks under Schottky groups. Take $k \geq 1$ pairs of mutually disjoint closed disks $\{(D_i, D'_i) : 1 \leq i \leq k\}$ in \mathbb{C} and choose Möbius transformations γ_i which maps D_i and D'_i and sends the interior of D_i to the exterior of D'_i , respectively. The group, say, Γ , generated by $\{\gamma_i : 1 \leq i \leq k\}$ is called a Schottky group of genus k (cf. [42, Sec. 2.7]). The Γ -orbits of the disks nest down onto the limit set $\Lambda(\Gamma)$ which is totally disconnected. If we denote by \mathcal{P} the union $\cup_{i=1}^k (\Gamma(C_i) \cup \Gamma(C'_i))$ where C_i and C'_i are the boundaries of D_i and D'_i respectively, \mathcal{P} is locally finite, as the nesting disks will become smaller and smaller (cf. [50, 4.5]). The common exterior of hemispheres above the initial disks D_i and D'_i , $1 \leq i \leq k$, is a fundamental domain for Γ in the upper half-space model \mathbb{H}^3 , and hence Γ is geometrically finite. Since \mathcal{P} consists of disjoint circles, Theorem 3.3 applies to \mathcal{P} . For instance, see Fig. 6 ([50, Fig. 4.11]). One can find many more explicit circle packings in [50] to which Theorem 3.3 applies.

4. Circle packings on the sphere

In the unit sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ with the Riemannian metric induced from \mathbb{R}^3 , the distance between two points is simply the angle between the rays connecting them to the origin $o = (0, 0, 0)$.

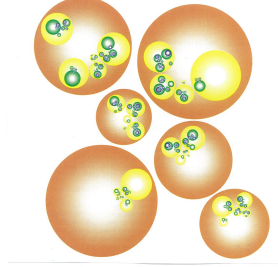


Figure 6. Schottky dance (reproduced with permission from Indra's Pearls, by D.Mumford, C. Series and D. Wright, copyright Cambridge University Press 2002)

Let \mathcal{P} be a circle packing on the sphere \mathbb{S}^2 , i.e., a union of circles. The spherical curvature of a circle C in \mathbb{S}^2 is given by

$$\text{Curv}_S(C) = \cot \theta(C)$$

where $0 < \theta(C) \leq \pi/2$ is the spherical radius of C , that is, the half of the visual angle of C from the origin o . We suppose that \mathcal{P} is infinite and locally finite in the sense that there are only finitely many circles in \mathcal{P} of spherical curvature at most T for any fixed $T > 0$.

For a region E of \mathbb{S}^2 , we set

$$N_T(\mathcal{P}, E) := \#\{C \in \mathcal{P} : C \cap E \neq \emptyset, \text{Curv}_S(C) < T\}.$$

We consider the Poincare ball model $\mathbb{B} = \{x_1^2 + x_2^2 + x_3^2 < 1\}$ of the hyperbolic 3 space with the metric d given by $\frac{2\sqrt{dx_1^2 + dx_2^2 + dx_3^2}}{1 - (x_1^2 + x_2^2 + x_3^2)}$. Note that the geometric boundary of \mathbb{B} is \mathbb{S}^2 and that for any circle C in \mathbb{S}^2 , we have

$$\sin \theta(C) = \frac{1}{\cosh d(\hat{C}, o)}$$

where $\hat{C} \subset \mathbb{B}$ is the convex hull of C . As both $\sin \theta$ and $\cosh d$ are monotone functions for $0 \leq \theta \leq \pi/2$ and $d \geq 0$ respectively, understanding $N_T(\mathcal{P}, E)$ is equivalent to investigating the number of Euclidean hemispheres on \mathbb{B} meeting the ball of hyperbolic radius T based at o whose boundaries are in \mathcal{P} and intersect E .

Let G denote the orientation preserving isometry group of \mathbb{B} .

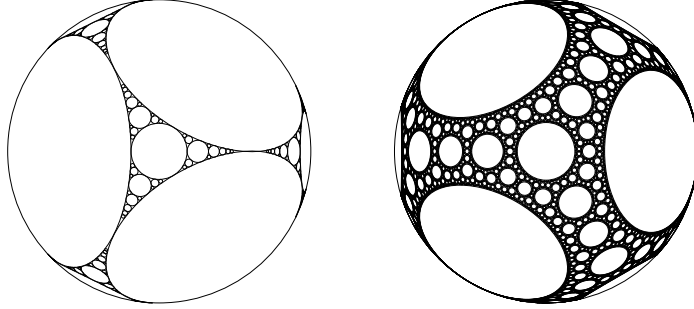


Figure 7. Apollonian packing and Sierpinski curve on the sphere (by C. McMullen)

Theorem 4.1 ([52]). *Let Γ be a non-elementary geometrically finite discrete subgroup of G and $\mathcal{P} = \cup_{i \in I} \Gamma(C_i)$ be an infinite, locally finite, and Γ -invariant circle packing on the sphere \mathbb{S}^2 with finitely many Γ -orbits.*

Suppose one of the following conditions hold:

1. Γ is convex co-compact;
2. all circles in \mathcal{P} are mutually disjoint;
3. $\cup_{i \in I} C_i^\circ \subset \Omega(\Gamma)$ where C_i° denotes the interior of C_i .

Then for any Borel subset $E \subset \mathbb{S}^2$ whose boundary is of zero Patterson-Sullivan measure,

$$N_T(\mathcal{P}, E) \sim \frac{\text{sk}_\Gamma(\mathcal{P}) \cdot \nu_o(E)}{\delta_\Gamma \cdot |m_\Gamma^{\text{BMS}}|} \cdot (2T)^{\delta_\Gamma} \quad \text{as } T \rightarrow \infty$$

where $0 < \text{sk}_\Gamma(\mathcal{P}) < \infty$ is defined in Def. 3.2.

5. Integral Apollonian packings: Primes and Twin primes

A circle packing \mathcal{P} is called *integral* if the curvatures of all circles in \mathcal{P} are integral. One of the special features of Apollonian circle packings is the abundant existence of *integral Apollonian circle packings*.

Descartes noted in 1643 (see [10]) that a quadruple (a, b, c, d) of real numbers can be realized as curvatures of four mutually tangent circles in the plane (oriented

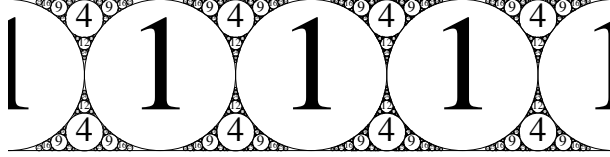


Figure 8. An Apollonian circle packing between two parallel lines.

so that their interiors are disjoint) if and only if it satisfies

$$2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = 0. \quad (5.1)$$

Usually referred to as the Descartes circle theorem, this theorem implies that if the initial four circles in an Apollonian circle packing \mathcal{P} in the plane have integral curvatures, then \mathcal{P} is an integral packing, as observed by Soddy in 1937 [70]. The Descartes circle theorem provides an integral Apollonian packing for every integral solution of the quadratic equation (5.1) and indeed there are infinitely many distinct integral Apollonian circle packings.

Let \mathcal{P} be an integral Apollonian circle packing. We can deduce from the existence of the lower bound for the non-zero curvatures in \mathcal{P} that such \mathcal{P} is either bounded or lies between two parallel lines. We assume that \mathcal{P} is primitive, that is, the greatest common divisor of curvatures is one.

Calling a circle with a prime curvature a prime and a pair of tangent prime circles a twin prime, Sarnak showed:

Theorem 5.2 ([64]). *There are infinitely many primes, as well as twin primes, in \mathcal{P} .*

For \mathcal{P} bounded, denote by $\pi^{\mathcal{P}}(T)$ the number of prime circles in \mathcal{P} of curvature at most T , and by $\pi_2^{\mathcal{P}}(T)$ the number of twin prime circles in \mathcal{P} of curvatures at most T . For \mathcal{P} congruent to the packing in Fig. 8, we alter the definition of $\pi^{\mathcal{P}}(T)$ and $\pi_2^{\mathcal{P}}(T)$ to count prime circles in a fixed period. Sarnak showed [64] that

$$\pi^{\mathcal{P}}(T) \gg \frac{T}{(\log T)^{3/2}}.$$

Recently Bourgain, Gamburd and Sarnak ([3] and [4]) obtained a uniform spectral gap for the family of congruence subgroups $\Gamma(q) = \{\gamma \in \Gamma : \gamma \equiv 1 \pmod{q}\}$, q square-free, of any finitely generated subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ provided $\delta_\Gamma > 1/2$. This theorem extends to a Zariski dense subgroup Γ of $\mathrm{SL}_2(\mathbb{Z}[i])$ and its congruence subgroups over square free ideals of $\mathbb{Z}[i]$ if $\delta_\Gamma > 1$.

Denoting by Q the Descartes quadratic form

$$Q(a, b, c, d) = 2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2,$$

the approach in [37] for counting circles in Apollonian circle packings which are either bounded or between two parallel lines is based on the interpretation of such circle counting problem into the counting problem for $w\Gamma \cap B_T^{\max}$ where $\Gamma < O_Q(\mathbb{Z})$ is the so-called Apollonian group, $w \in \mathbb{Z}^4$ with $Q(w) = 0$ and B_T^{\max} denotes the maximum norm ball in \mathbb{R}^4 .

Using the spin double cover $\text{Spin}_Q \rightarrow \text{SO}_Q$ and the isomorphism $\text{Spin}_Q(\mathbb{R}) = \text{SL}_2(\mathbb{C})$, we use the aforementioned result of Bourgain, Gamburd and Sarnak to obtain a smoothed counting for $w\Gamma(q) \cap B_T$ with a uniform error term for the family of square-free congruence subgroups $\Gamma(q)$'s where B_T is the Euclidean norm ball. This is a crucial ingredient for the Selberg's upper bound sieve, which is used to prove the following:

Theorem 5.3 ([37]). *As $T \rightarrow \infty$,*

$$\pi^{\mathcal{P}}(T) \ll \frac{T^\alpha}{\log T}, \quad \text{and} \quad \pi_2^{\mathcal{P}}(T) \ll \frac{T^\alpha}{(\log T)^2}$$

where $\alpha = 1.30568(8)$ is the residual dimension of \mathcal{P} .

Remark 5.4. 1. Modulo 16, the Descartes equation (5.1) has no solutions unless two of the curvatures are even and the other two odd. In particular, there are no “triplet primes” of three mutually tangent circles, all having odd prime curvatures.

2. We can also use the methods in [37] to give lower bounds for almost primes in a packing. A circle in \mathcal{P} is called *R-almost prime* if its curvature is the product of at most R primes. Similarly, a pair of tangent circles is called *R-almost twin prime* if both circles are *R-almost prime*. Employing Brun's combinatorial sieve, our methods show the existence of $R_1, R_2 > 0$ (unspecified) such that the number of R_1 -almost prime circles in \mathcal{P} whose curvature is at most T is $\asymp \frac{T^\alpha}{\log T}$,¹ and that the number of pairs of R_2 -almost twin prime circles whose curvatures are at most T is $\asymp \frac{T^\alpha}{(\log T)^2}$.

3. A suitably modified version of Conjecture 1.4 in [3], a generalization of Schinzel's hypothesis, implies that for some $c, c_2 > 0$,

$$\pi^{\mathcal{P}}(T) \sim c \cdot \frac{T^\alpha}{\log T} \quad \text{and} \quad \pi_2^{\mathcal{P}}(T) \sim c_2 \cdot \frac{T^\alpha}{(\log T)^2}.$$

The constants c and c_2 are detailed in [24].

4. Recently Bourgain and Fuchs [2] showed that in a given bounded integral Apollonian packing \mathcal{P} , the growth of the number of *distinct* curvatures at most T is at least $c \cdot T$ for some $c > 0$.

¹ By $f(T) \asymp g(T)$, we mean $g(T) \ll f(T) \ll g(T)$.

5. The spherical Soddy-Gossett theorem says (see [38]) that the quadruple (a, b, c, d) of spherical curvatures of four mutually tangent circles in \mathcal{P} satisfies

$$2(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2 = -4.$$

This theorem implies again that there are infinitely many *integral* spherical Apollonian circle packings, that is, the spherical curvature of every circle is integral. It will be interesting to have results analogous to Theorems 5.2 and 5.3 for integral spherical Apollonian packings.

6. Equidistribution in geometrically finite hyperbolic manifolds

Let G be the identity component of the group of isometries of \mathbb{H}^n and $\Gamma < G$ be a non-elementary geometrically finite discrete subgroup.

We have discussed that the Bowen-Margulis-Sullivan measure is a finite measure on the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^n)$ which is mixing for the geodesic flow. Another measure playing an important role in studying the dynamics of flows on $T^1(\Gamma \backslash \mathbb{H}^n)$ is the following Burger-Roblin measure.

Burger-Roblin measure: The Burger-Roblin measure m_Γ^{BR} is the induced measure on $T^1(\Gamma \backslash \mathbb{H}^n)$ of the following Γ -invariant measure on $T^1(\mathbb{H}^n)$:

$$d\tilde{m}^{\text{BR}}(u) = e^{(n-1)\beta_{u^+}(x, \pi(u))} e^{\delta_\Gamma \beta_{u^-}(x, \pi(u))} dm_x(u^+) d\nu_x(u^-) dt$$

where m_x denotes the probability measure on the boundary $\partial_\infty(\mathbb{H}^n)$ invariant under the maximal compact subgroup $\text{Stab}_G(x_0)$. For any x and $x_0 \in \mathbb{H}^n$, we have $dm_x(\xi) := e^{-(n-1)\beta_\xi(x, x_0)} dm_{x_0}(\xi)$ and it follows that this definition of m_Γ^{BR} is independent of the choice of $x \in \mathbb{H}^n$.

Burger [8] showed that for a convex cocompact hyperbolic surface with δ_Γ at least $1/2$, this is a unique ergodic horocycle invariant measure up to homothety. Roblin [59] extended Burger's result in much greater generality, for instance, including all non-elementary geometrically finite hyperbolic manifolds.

The name of the Burger-Roblin measure was first suggested by Shah and the author in [37] and [53] in recognition of this important classification result.

We note that the total mass $|m_\Gamma^{\text{BR}}|$ is finite only when $\delta_\Gamma = n-1$ (or equivalently only when Γ is a lattice in G) and is supported on the set $\{u \in T^1(\Gamma \backslash \mathbb{H}^n) : u^- \in \Lambda(\Gamma)\}$.

Let $S^\dagger \subset T^1(\mathbb{H}^n)$ be one of the following:

1. an unstable horosphere;
2. the oriented unit normal bundle of a codimension one totally geodesic subspace of \mathbb{H}^n

3. the set of outward normal vectors to a (hyperbolic) sphere in \mathbb{H}^n .

We consider the following measures on $\text{Stab}_\Gamma(S^\dagger) \backslash S^\dagger$:

$$d\mu_{S^\dagger}^{\text{Leb}}(s) = e^{(n-1)\beta_{s^+}(x, \pi(s))} dm_x(s^+), \quad d\mu_{S^\dagger}^{\text{PS}}(s) = e^{\delta_\Gamma \beta_{s^+}(x, \pi(s))} d\nu_x(s^+)$$

for any $x \in \mathbb{H}^n$.

Denote by p the canonical projection $T^1(\mathbb{H}^n) \rightarrow T^1(\Gamma \backslash \mathbb{H}^n) = \Gamma \backslash T^1(\mathbb{H}^n)$.

Theorem 6.1 ([53]). *For $\psi \in C_c(T^1(\Gamma \backslash \mathbb{H}^n))$ and any relatively compact subset $\mathcal{O} \subset p(S^\dagger)$ with $\mu_{S^\dagger}^{\text{PS}}(\partial(\mathcal{O})) = 0$,*

$$e^{(n-1-\delta_\Gamma)t} \cdot \int_{\mathcal{O}} \psi(g^t(s)) d\mu_{S^\dagger}^{\text{Leb}}(s) \sim \frac{\mu_{S^\dagger}^{\text{PS}}(\mathcal{O}_*)}{\delta_\Gamma \cdot |m_\Gamma^{\text{BMS}}|} \cdot m_\Gamma^{\text{BR}}(\psi) \quad \text{as } t \rightarrow \infty$$

where

$$\mathcal{O}_* = \{s \in \mathcal{O} : s^+ \in \Lambda(\Gamma)\}.$$

Definition 6.2. For a hyperbolic subspace $S = \mathbb{H}^{n-1} \subset \mathbb{H}^n$, we say that a parabolic fixed point $\xi \in \Lambda(\Gamma) \cap \partial_\infty(\mathbb{H}^{n-1})$ of Γ is *internal* if any parabolic element $\gamma \in \Gamma$ fixing ξ preserves \mathbb{H}^{n-1} .

Recalling the notation π for the canonical projection from $T^1(\mathbb{H}^n)$ to \mathbb{H}^n , we set $S = \pi(S^\dagger)$.

Theorem 6.3 ([53]). *We assume that the projection map $\text{Stab}_\Gamma(S) \backslash S \rightarrow \Gamma \backslash \mathbb{H}^n$ is proper. In the case when S is a codimension one totally geodesic subspace, we also assume that every parabolic fixed point of Γ in the boundary of S is internal.*

For $\psi \in C_c(T^1(\Gamma \backslash \mathbb{H}^n))$,

$$e^{(n-1-\delta_\Gamma)t} \cdot \int_{p(S^\dagger)} \psi(g^t(s)) d\mu_{S^\dagger}^{\text{Leb}}(s) \sim \frac{\mu_{S^\dagger}^{\text{PS}}(S_*^\dagger)}{\delta_\Gamma \cdot |m_\Gamma^{\text{BMS}}|} \cdot m_\Gamma^{\text{BR}}(\psi) \quad \text{as } t \rightarrow \infty$$

where

$$S_*^\dagger = \{s \in p(S^\dagger) : s^+ \in \Lambda(\Gamma)\}.$$

We have $0 \leq \mu_{S^\dagger}^{\text{PS}}(S_^\dagger) < \infty$, and $\mu_{S^\dagger}^{\text{PS}}(S_*^\dagger) = 0$ may happen only when S is totally geodesic.*

It can be shown by combining results of [11] and [45] that in a finite volume space $\Gamma \backslash \mathbb{H}^n$, the properness of the projection map $\text{Stab}_\Gamma(S) \backslash S \rightarrow \Gamma \backslash \mathbb{H}^n$ implies that $\text{Stab}_\Gamma(S) \backslash S$ is of finite volume as well, except for the case when $n = 2$ and S is a proper geodesic in \mathbb{H}^2 connecting two parabolic fixed points of a lattice $\Gamma < \text{PSL}_2(\mathbb{R})$.

When both $\Gamma \backslash \mathbb{H}^n$ and $\text{Stab}_\Gamma(S) \backslash S$ are of finite volume, we have $n - 1 = \delta_\Gamma$ and both m_Γ^{BMS} and m_Γ^{BR} are finite invariant measures and $\mu_{S^\dagger}^{\text{PS}} = \mu_{S^\dagger}^{\text{Leb}}$ (up to a constant multiple). In this case, Theorem 6.3 is due to Sarnak [63] for the closed horocycles for $n = 2$. The general case is due to Duke, Rudnick and Sarnak [14] and Eskin and McMullen [19] gave a simpler proof of Theorem 6.3, based on the

mixing property of the geodesic flow of a finite volume hyperbolic manifold. The latter proof, combined with a strengthened version of the wavefront lemma [28], also works for proving Theorem 6.1. We remark that the idea of using mixing in this type of problem goes back to the 1970 thesis of Margulis [46] (see also [31, Appendix]). Eskin, Mozes and Shah [20] and Shah [69] provided yet another different proofs using the theory of unipotent flows. When both $\Gamma \backslash \mathbb{H}^n$ and $\text{Stab}_\Gamma(S) \backslash S$ are of finite volume, Theorem 6.1 easily implies Theorem 6.3 but not conversely.

In the case when S^\dagger is a horosphere, Theorem 6.1 was obtained in [59], and Theorem 6.3 was proved in [37] when $\delta_\Gamma > (n-1)/2$ with a different interpretation of the main term.

Remark 6.4. 1. The condition on the internality of all parabolic fixed points of Γ in the boundary of S is crucial, as $\mu_{S^\dagger}^{\text{PS}}(S_*^\dagger) = \infty$ otherwise. This can already be seen in the level of a lattice: take $\Gamma = \text{SL}_2(\mathbb{Z})$ and let S be the geodesic connecting 0 and ∞ in the upper half space model. Then any upper triangular matrix in Γ fixes ∞ but does not stabilize S . Indeed the length of the image of S in $\Gamma \backslash \mathbb{H}^2$ is infinite.

2. In proving Theorem 3.3, we count circles in \mathbb{C} by counting the corresponding Euclidean hemispheres in \mathbb{H}^3 . As the Euclidean hemispheres are totally geodesic hyperbolic planes, this amounts to understanding the distribution of a Γ -orbit of a totally geodesic hyperbolic plane in \mathbb{H}^3 . The equidistribution theorem we use here is Theorem 6.3 for S a hyperbolic plane.

3. More classical applications of the equidistribution theorem such as Theorem 6.3 can be found in the point counting problems of Γ -orbits in various spaces. For a Γ -orbit in the hyperbolic space \mathbb{H}^n , the orbital counting in Riemannian balls was obtained Lax-Phillips [39] for $\delta_\Gamma > \frac{n-1}{2}$ and by Roblin [59] in general.

Extending the work of Duke, Rudnick and Sarnak [14] and of Eskin and McMullen [19] for Γ lattices, we obtain in [53], for any geometrically finite group Γ of G , the asymptotic of the number of vectors of norm at most T lying in a discrete orbit $w\Gamma$ of a quadric

$$F(x_1, \dots, x_{n+1}) = y$$

for a real quadratic form F of signature $(n, 1)$ and any $y \in \mathbb{R}$ (when $y > 0$, there is an extra assumption on w not being Γ strongly parabolic. See [53] for details). When $y = 0$ and $n = 2, 3$, special cases of this result were obtained in [35], [37] and [36] under the condition $\delta_\Gamma > (n-1)/2$. Based on the Descartes circle theorem, this result in [37] was used to prove Theorem 3.5 for the bounded Apollonian packings. In [40], a Γ -orbit in the geometric boundary is shown to be equidistributed with respect to the Patterson-Sullivan measure, extending the work [25] for the lattice case.

4. For $\psi \in C_c(\Gamma \backslash \mathbb{H}^n)$, we have

$$m_\Gamma^{\text{BR}}(\psi) = \langle \psi, \phi_\Gamma \rangle := \int_{\Gamma \backslash \mathbb{H}^n} \psi(x) \cdot \phi_\Gamma(x) \, dm^{\text{Leb}}(x)$$

where $\phi_\Gamma(x) = |\nu_x|$ is the positive eigenfunction of the Laplace operator on $\Gamma \backslash \mathbb{H}^n$ with eigenvalue $\delta_\Gamma(n-1-\delta_\Gamma)$ and

$$dm^{\text{Leb}}(u) = e^{(n-1)\beta_{u^+}(x, \pi(u))} e^{(n-1)\beta_{u^-}(x, \pi(u))} dm_x(u^+) dm_x(u^-) dt$$

for any $x \in \mathbb{H}^n$. Hence Theorem 6.3 says that for $\psi \in C_c(\Gamma \backslash \mathbb{H}^n)$,

$$e^{(n-1-\delta_\Gamma)t} \cdot \int_{p(S^\dagger)} \psi(\pi(g^t(s))) d\mu_{S^\dagger}^{\text{Leb}}(s) \sim \frac{\mu_{S^\dagger}^{\text{PS}}(S_*^\dagger)}{\delta_\Gamma \cdot |m_\Gamma^{\text{BMS}}|} \cdot \langle \psi, \phi_\Gamma \rangle \quad \text{as } t \rightarrow \infty. \quad (6.5)$$

When $\delta_\Gamma > (n-1)/2$, $\phi_\Gamma \in L^2(\Gamma \backslash \mathbb{H}^n)$ and its eigenvalue $\delta_\Gamma(n-1-\delta_\Gamma)$ is isolated in the L^2 -spectrum of the Laplace operator [39]. It will be desirable to obtain a rate of convergence in (6.5) in terms of the spectral gap of Γ in such cases. For Γ lattices, it was achieved in [14] for $p(S)$ compact and in [1] in general. This was done in the case of a horosphere in [37], which was the main ingredient in the proof of Theorem 5.3. It may be possible to extend the methods of [37] to obtain an error term in general.

7. Further remarks and questions

Let G be the identity component of the group of isometries of \mathbb{H}^n and Γ be a geometrically finite group. We further assume that Γ is Zariski dense in G for discussions in this section. When we identify \mathbb{H}^n with G/K for a maximal compact subgroup K , the unit tangent bundle $T^1(\mathbb{H}^n)$ can be identified with G/M where M is the centralizer in K of a Cartan subgroup, say, A , whose multiplication on the right corresponds to the geodesic flow. The frame bundle of \mathbb{H}^n can be identified with G and the frame flow on the frame bundle is given by the multiplications by elements of A on the right.

We have stated the equidistribution results in section 6 in the level of the unit tangent bundle $T^1(\Gamma \backslash \mathbb{H}^n)$. As the frame bundle is a homogeneous space of G unlike the unit tangent bundle, it is much more convenient to work in the frame bundle. Fortunately, as observed in [23], the frame flow is mixing on $\Gamma \backslash G$ with respect to the lift from $\Gamma \backslash G/M$ to $\Gamma \backslash G$ of the Bowen-Margulis-Sullivan measure. Using this, we can extend Theorems 6.1 and 6.3 to the level of the frame bundle $\Gamma \backslash G$. It seems that the classification theorem of Burger and Roblin can also be extended: for a horospherical group N , any locally finite N -invariant ergodic measure on $\Gamma \backslash G$ is either supported on a closed N -orbit or the lift of the Burger-Roblin measure (we caution here that a locally finite N -invariant measure supported on a closed N -orbit need not be a finite measure unlike the Γ -lattice cases).

In analogy with Ratner's theorem [56], [57], we propose the following problems: let U be a one-parameter unipotent subgroup, or more generally a subgroup generated by unipotent one parameter subgroups of G :

1. [Measure rigidity] Classify all locally finite Borel U -invariant ergodic measures on $\Gamma \backslash G$.

2. [Topological rigidity] Classify the closures of U -orbits in $\Gamma \backslash G$.

We remark that as $G = \mathrm{SO}(n, 1)$ (up to a local isomorphism) in our set-up, the above topological rigidity for Γ lattices was also obtained by Shah ([68], [67]) based on the approach of Margulis ([43], [44]) and of Dani and Margulis [13].

Both questions are known for $n = 2$ due to Burger [8] and Roblin [59], as in this case, there is only one unipotent one-parameter subgroup up to conjugation, which gives the horocycle flow. Shapira used them to prove equidistribution for non-closed horocycles [65].

It may be a good idea to start with a sampling case when $G = \mathrm{SL}_2(\mathbb{C})$, $U = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ Zariski dense and geometrically finite.

1. Are there any locally finite $\mathrm{SL}_2(\mathbb{R})$ -invariant ergodic measure on $\Gamma \backslash \mathrm{SL}_2(\mathbb{C})$ besides the Haar measure (=the $\mathrm{SL}_2(\mathbb{C})$ -invariant measures) and the $\mathrm{SL}_2(\mathbb{R})$ -invariant measures supported on closed $\mathrm{SL}_2(\mathbb{R})$ orbits?
2. Is every non-closed $\mathrm{SL}_2(\mathbb{R})$ -orbit dense in $\Gamma \backslash \mathrm{SL}_2(\mathbb{C})$?

It seems that the answers are *no* for (1) and *yes* for (2).

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